Numerical Study of Algebraic Solutions to Linear Problems Involving Stochastic Parameters

Rene Alt[1], Jean-Luc Lamotte[1], and Svetoslav Markov[2]

[1] Laboratoire Informatique Paris 6, U. Pierre et Marie Curie,
4 place Jussieu, 75252 Paris cedex 05, France
{Rene.Alt, Jean-Luc.Lamotte}@lip6.fr

[2] Institute of Mathematics and Informatics, BAS,
"Acad. G. Bonchev" st., block 8, 1113 Sofia, Bulgaria
smarkov@bio.bas.bg

Abstract. We formulate certain numerical problems with stochastic numbers and compare algebraically obtained results with experimental results provided by the CESTAC method. Such comparisons give additional information related to the stochastic behavior of random roundings in the course of numerical computations. The good coincidence between theoretical and experimental results confirms the adequacy of our algebraic model and its possible application in the numerical practice.

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1 Introduction

Stochastic numbers are gaussian random variables with a known mean value and a known standard deviation. Some fundamental properties of stochastic numbers are considered in [9]. The mean values of the stochastic numbers satisfy the usual real arithmetic, whereas standard deviations are added and multiplied by scalars in a specific way. As regard to addition standard deviations form an abelian monoid with cancellation law. This monoid can be embedded in an additive group and after a suitable extension of multiplication by scalars one obtains a so-called s-space, which is in fact a vector space with a specifically defined multiplication by scalar [2], [4]. This allows us to introduce in s-spaces concepts like linear combination, basis, dimension etc. Thus, in theory, computations in s-spaces are reduced to computations in vector spaces. This opens the road to finding explicit expressions for the solution of certain algebraic problems involving stochastic numbers.

Alternatively, stochastic numbers can be computed experimentally using the CESTAC method, which is a Monte-Carlo method consisting in performing each arithmetic operation several times using an arithmetic with a random rounding mode [3], [7], [8]. For a survey of methods using Monte-Carlo arithmetic see [6].

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In Sections 2 we briefly present the main results of our theory of s-spaces as regard to the arithmetic operations for addition and multiplication by scalars needed for the purposes of this study; for a detailed presentation of the theory, see [4]. Section 3 considers the algebraic solution of linear systems of equations with right-hand sides involving stochastic numbers. In Section 4 we extend further our idea from [5] to compare the theoretic solution of an algebraic problem involving stochastic numbers with the solution obtained numerically by the CESTAC method. Numerical experiments are reported and a good coincidence between theoretical and experimental results is observed.

2 Stochastic Numbers and Stochastic Arithmetic

By $\mathbb{R}$ we denote the set of reals; the same notation is used for the linearly ordered field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. For any integer $n \geq 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space under the familiar operations of addition and multiplication by scalars denoted by $\mathbf{V}^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$, $n \geq 1$. By $\mathbb{R}^+$ we denote the set of nonnegative reals.

2.1 The Arithmetic for Stochastic Numbers

A *stochastic number* $X = (m; s)$ is a gaussian random variable with mean value $m \in \mathbb{R}$ and (nonnegative) standard deviation $s \in \mathbb{R}^+$. The set of all stochastic numbers is $\mathbb{S} = \{(m; s) \mid m \in \mathbb{R}, s \in \mathbb{R}^+\}$. Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$. Addition and multiplication by scalars are defined by:

$$X_1 + X_2 = (m_1; s_1) + (m_2; s_2) \stackrel{def}{=} \left(m_1 + m_2;\ \sqrt{s_1^2 + s_2^2} \right),$$

$$\gamma * X = \gamma * (m; s) \stackrel{def}{=} \left(\gamma m;\ |\gamma| s \right),\ \gamma \in \mathbb{R}.$$

It has to be noticed that the operations on stochastic numbers are error free and are only used for theory. In this approach stochastic numbers are only used as a model for computation on data containing errors.

A stochastic number of the form $(0; s)$, $s \in \mathbb{R}^+$, is called *(centrally) symmetric*. If X_1, X_2 are symmetric stochastic numbers, then $X_1 + X_2$ and $\lambda * X_1$, $\lambda \in \mathbb{R}$, are also symmetric stochastic numbers. Thus there is a 1–1 correspondence between the set of symmetric stochastic numbers and the set $\mathbb{R}^+$. We shall use special symbols "$\oplus$", "$*$" for the arithmetic operations over standard deviations, as these operations are different from the corresponding ones for numbers. The operations "$\oplus$", "$*$" induce a special arithmetic on the set $\mathbb{R}^+$. Consider the system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$, such that for $s, t \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$:

$$s \oplus t = \sqrt{s^2 + t^2}, \quad \gamma * s = |\gamma| s. \tag{1}$$

Proposition 1. *[4] The system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$ is an abelian additive monoid with cancellation, such that for $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$:*

$$\alpha * (s \oplus t) = \alpha * s \oplus \alpha * t, \tag{2}$$
$$\alpha * (\beta * s) = (\alpha\beta) * s, \tag{3}$$
$$1 * s = s, \tag{4}$$
$$(-1) * s = s, \tag{5}$$
$$\sqrt{\alpha^2 + \beta^2} * s = \alpha * s \oplus \beta * s, \ \alpha, \beta \geq 0. \tag{6}$$

More generally, we can extend componentwise operations (1) for n-tuples $s = (s_1, ..., s_n)$, $s_i, \in \mathbb{R}^+$, that is,

$$(s_1, ..., s_k) \oplus (t_1, ..., t_k) = (s_1 \oplus t_1, ..., s_k \oplus t_k), \tag{7}$$
$$\gamma * (s_1, s_2, ..., s_k) = (|\gamma| s_1, |\gamma| s_2, ..., |\gamma| s_k), \ \gamma \in \mathbb{R}. \tag{8}$$

The corresponding system $((\mathbb{R}^+)^n, \oplus, \mathbb{R}, *)$ again satisfies the conditions of Proposition 1. A system satisfying the conditions of Proposition 1 is called an *s-space of monoid structure*. Such a structure can be naturally embedded into a group, obtaining thus an *s-space of group structure*, as shown below.

2.2 The S-Space of Group Structure

For $s \in \mathbb{R}$ denote $\tau(s) = \{+, \ \text{if} \ \ s \geq 0; \ -, \ \text{if} \ \ s < 0\}$. We extend the operation addition "$\oplus$" for all $s, t \in \mathbb{R}$, admitting thus negative reals, corresponding to *improper* standard deviations:

$$s \oplus t \overset{def}{=} \tau(s+t)\sqrt{|\tau(s)s^2 + \tau(t)t^2|}. \tag{9}$$

We note that $\tau(s+t) = \tau(\tau(s)s^2 + \tau(t)t^2) = \tau(s \oplus t)$ for $s, t \in \mathbb{R}$. Using (9) we embed isomorphically the monoid $(\mathbb{R}^+, \oplus)$ into the system $(\mathbb{R}, \oplus)$, which is an abelian group with null 0 and opposite element $\text{opp}(s) = -s$, i. e. $s \oplus (-s) = 0$. Indeed, from (9) we have $s \oplus (-s) = \tau(s-s)\sqrt{|\tau(s)s^2 - \tau(s)s^2|} = \tau(0)\sqrt{0} = 0$. Here are some examples of addition in the system $(\mathbb{R}, \oplus)$: $1 \oplus 1 = \sqrt{2}$, $1 \oplus 2 = \sqrt{5}$, $3 \oplus 4 = 5$, $4 \oplus (-3) = \sqrt{7}$, $3 \oplus (-4) = -\sqrt{7}$, $5 \oplus (-4) = 3$, $4 \oplus (-5) = -3$, $(-3) \oplus (-4) = -5$, $1 \oplus 2 \oplus 3 = \sqrt{14}$, $1 \oplus 2 \oplus (-3) = -2$.

Using (9) and $\tau(s_1 \oplus ... \oplus s_n) = \tau(s_1 + ... + s_n)$ we obtain for $n \geq 2$

$$s_1 \oplus s_2 \oplus ... \oplus s_n = \tau(s_1 + ... + s_n)\sqrt{|\tau(s_1)s_1^2 + ... + \tau(s_n)s_n^2|}. \tag{10}$$

Proposition 2. *For $s_1, s_2, ..., s_n, t \in \mathbb{R}$ the equation $s_1 \oplus s_2 \oplus ... \oplus s_n = t$ is equivalent to $\tau(s_1)s_1^2 + ... + \tau(s_n)s_n^2 = \tau(t)t^2$.*

The proof follows immediately from the fact that the equation $\tau(s)\sqrt{|s|} = t$ implies $s = \tau(t)t^2$, and, in particular, $\tau(t) = \tau(s)$.

Multiplication by scalars is naturally extended on the set $\mathbb{R}$ of generalized standard deviations by: $\gamma * s = |\gamma| s$, $s \in \mathbb{R}$. Multiplication by -1 (negation) is $(-1) * s = |-1| s = s$, $s \in \mathbb{R}$, in accordance with (4)–(5). To avoid confusion we shall write the scalars always to the left side of the standard deviation. Under this convention we have, e. g. $(-2) * 2 = 4$, whereas $2 * (-2) = -4$.

Note that if s is a standard deviation, then we have $\gamma * s = (-\gamma) * s$ for any $\gamma \in \mathbb{R}$; thus multiplication by negative scalar does not change the type of s (proper/improper).

It is easy to check that all conditions (2)–(6) of Proposition 1 hold true for generalized standard deviations. This justifies the following definition:

Definition 1. *A system $(\mathcal{S}, \oplus, \mathbb{R}, *)$, such that: i) $(\mathcal{S}, \oplus)$ is an abelian additive group, and, ii) for any $s, t \in \mathcal{S}$, $\alpha, \beta \in \mathbb{R}$ relations (2)–(6) hold, is called an s-space over $\mathbb{R}$ (with group structure).*

3 Linear Systems with Stochastic Right-Hand Side

3.1 Canonical S-Spaces and Dot Product

For any integer $k \geq 1$ the set $\mathcal{S} = \mathbb{R}^k$ of all k-tuples $(s_1, s_2, ..., s_k)$ forms an s-space over $\mathbb{R}$ under the operations (7)–(8), whenever the sums $s_i \oplus t_i$ in (7) are defined by (9). The s-space $\mathbf{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ is the *canonical s-space (of standard deviations)*. In the s-space $\mathbf{S}^k$ we introduce a scalar (dot) product. Namely, for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{R}^k, s = (s_1, s_2, ..., s_k) \in \mathbf{S}^k$ we define $\alpha * s = \alpha_1 * s_1 \oplus \alpha_2 * s_2 \oplus ... \oplus \alpha_k * s_n$.

Using (10) we obtain for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{R}^k,\ s = (s_1, s_2, ..., s_k) \in \mathbf{S}^k$

$$\alpha * s = \alpha_1 * s_1 \oplus ... \oplus \alpha_k * s_k = \tau(\alpha * s)\sqrt{|\alpha_1^2 \tau(s_1) s_1^2 + ... + \alpha_k^2 \tau(s_k) s_k^2|}.$$

Example 1. Let $\alpha_i = 1, s_i = s, i = 1, ..., k$. Then $\alpha * s = s \oplus ..._{(k\ times)} \oplus s = \tau(s)\sqrt{ks^2} = s\sqrt{k}$. This fact has been already known for long [1].

Proposition 3. *For $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{R}^k, (s_1, s_2, ..., s_k) \in \mathbf{S}^k$ the equation $\alpha * s = t$ is equivalent to $\alpha_1^2 \tau(s_1) s_1^2 + ... + \alpha_k^2 \tau(s_k) s_k^2 = \tau(t) t^2$.*

Remark. It is used in the proof that $\tau(\alpha_i * s_i) = \tau(s_i)$.

3.2 S-Spaces and Their Relation to Vector Spaces

Proposition 4. *Let $(\mathcal{S}, +, \mathbb{R}, *)$ be an s-space over $\mathbb{R}$. Then the system $(\mathcal{S}, +, \mathbb{R}, \cdot)$ where the operation "$\cdot$": $\mathbb{R} \times \mathcal{S} \longrightarrow \mathcal{S}$ is defined by*

$$\alpha \cdot c = \begin{cases} \sqrt{|\alpha|} * c, & \text{if } \alpha \geq 0; \\ \sqrt{|\alpha|} * (-c), & \text{if } \alpha < 0, \end{cases} \tag{11}$$

is a vector space over $\mathbb{R}$. Conversely, let $(\mathcal{S}, +, \mathbb{R}, \cdot)$ be a vector space over $\mathbb{R}$. The system $(\mathcal{S}, +, \mathbb{R}, *)$ is an s-space over $\mathbb{R}$ whenever "$*$" is defined by

$$\alpha * c = \alpha^2 \cdot c. \tag{12}$$

Proposition 4 shows that each one of the two *associted* spaces $(\mathcal{S}, +, \mathbb{R}, *)$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ can be obtained from the other one by a redefinition of the operation multiplication by scalars using (11), resp. (12). Assume that $\mathcal{S} = (\mathcal{S}, +, \mathbb{R}, *)$ is an s-space over $\mathbb{R}$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ is the associated vector space. All vector space concepts from the vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$, such as linear combination, linear dependence, basis etc., apply to the s-space $(\mathcal{S}, +, \mathbb{R}, *)$ [4].

Theoretically stochastic numbers are defined as elements of the direct sum $\mathcal{V} \oplus \mathcal{S}$ of a vector space $\mathcal{V}$ and a s-space $\mathcal{S}$ both of same dimension k. Namely, let $\mathcal{V} = \mathbf{V}^k$ be a k-dimensional vector space with a basis $(v^{(1)}, ..., v^{(k)})$ and let $\mathcal{S} = \mathbf{S}^k$ be a k-dimensional s-space having a basis $(s^{(1)}, ..., s^{(k)})$. Then $(v^{(1)}, ..., v^{(k)}; s^{(1)}, ..., s^{(k)})$ is a basis of the *k-dimensional* space $\mathbf{V}^k \oplus \mathbf{S}^k$. Such a setting allows us to consider numerical problems involving vectors and matrices, wherein certain numeric variables have been substituted by stochastic ones.

3.3 Stochastic Linear Systems

We consider a linear system $Ax = b$, such that A is a real $n \times n$-matrix and the right-hand side b is a vector of stochastic numbers. Then the solution x also consists of stochastic numbers, and, respectively, all arithmetic operations (additions and multiplications by scalars) in the expression Ax involve stochastic numbers; we denote this by writing $A * x$ instead of Ax.

Problem. Assume that $A = (\alpha_{ij})_{i,j=1}^{n}$, $\alpha_{ij} \in \mathbb{R}$, is a real $n \times n$-matrix, and $b = (b'; b'')$ is a n-tuple of (generalized) stochastic numbers, such that $b', b'' \in \mathbb{R}^n$, $b' = (b'_1, ..., b'_n)$, $b'' = (b''_1, ..., b''_n)$. We look for a (generalized) stochastic vector $x = (x'; x'')$, $x', x'' \in \mathbb{R}^n$, satisfying the system $A * x = b$.

Solution. Due to $A*x = A*(x'; x'') = (Ax'; A*x'')$ the system $A*x = b$ reduces to a linear system $Ax' = b'$ for the vector $x' = (x'_1, ..., x'_n)$ of mean values and a system $A * x'' = b''$ for the standard deviations $x'' = (x''_1, ..., x''_n)$. If $A = (\alpha_{ij})$ is nonsingular, then $x' = A^{-1}b'$. We shall next concentrate on the solution of the system $A * x'' = b''$ for the standard deviations.

The i-th equation of the system $A*x'' = b''$ reads $\alpha_{i1} * x''_1 \oplus ... \oplus \alpha_{in} * x''_n = b''_i$. According to Proposition 3, this is equivalent to

$$\alpha_{i1}^2 \tau(x''_1) x_1''^2 + ... + \alpha_{in}^2 \tau(x''_n) x_n''^2 = \tau(b''_i) b_i''^2, \quad i = 1, ..., n.$$

Setting $\tau(x''_i)(x''_i)^2 = y_i$, $\tau(b''_i)(b''_i)^2 = c_i$, we obtain a linear $n \times n$ system $Dy = c$ for $y = (y_i)$, where $D = (\alpha_{ij}^2)$, $c = (c_i)$. If D is nonsingular we can solve the system $Dy = c$ for the vector y, $y = D^{-1}c$, and then obtain the standard deviation vector x'' by means of $x''_i = \tau(y_i)\sqrt{|y_i|}$. Thus for the solution of the original problem it is necessary and sufficient that both matrices $A = (\alpha_{ij})$ and $D = (\alpha_{ij}^2)$ are nonsingular.

Summarizing, to solve $A * x = b$ we perform the following steps:

i) check the matrices $A = (\alpha_{ij})$ and $D = (\alpha_{ij}^2)$ for nonsingularity;
ii) find the solution $x' = A^{-1}b'$ of the linear system $Ax' = b'$;
iii) find the solution $y = D^{-1}c$ of the linear system $Dy = c$, where $c = (c_i)$, $c_i = \tau(b''_i)(b_i''^2)$. Compute $x''_i = \tau(y_i)\sqrt{|y_i|}$; then the solution of $A * x = b$ is $x = (x'; x'')$.

4 Numerical Experiments

Numerical experiments have been performed in order to compare the theoretical results with numerical results obtained by means of the CESTAC method for imprecise stochastic data.

Scalar Product. Let α be a real vector of size N with $\alpha_i = i,\ i = 1, ..., N$. Assume that b is a stochastic vector of size N. All samples for the components of b have been generated with a gaussian generator with mean value $m = 1$ and standard deviation $\sigma = 0.001$.

Theoretically, the standard deviation of the dot product $\alpha * b$ is equal to $\sigma\sqrt{N(N+1)(2N+1)/6}$. On the other hand, according to the theory of the CESTAC method, a stochastic number can be represented by an n-tuple of random values with a known mean value m and a known standard deviation σ. In our examples $n = 3$ as implemented in the CADNA software [3].

With the above conditions ($m = 1, \sigma = 0.001$) the scalar product $\alpha * b$ has been computed k times for various sizes $N = 10, 100, ..., 10000$. For each size N the mean value $\overline{\delta}$ of the standard deviation δ_i of the result ($i = 1, 2, ..., k$) has been computed.

This provides samples of size k whose mean values approximate the theoretical standard deviation.

Table 1 reports the percentages of cases where the theoretical standard deviation $\sigma\sqrt{N(N+1)(2N+1)/6}$ is outside the computed confidence interval. These percentages have been computed with 1000 runs.

Comments: From Table 1, it is clear a posteriori that the distribution of the scalar product is effectively gaussian, as a size of 4 to 5 for the samples is enough to approximate the theoretical value, whereas if this were not the case, then the samples should have rather be of size 30.

4.1 Solution of a Linear System $A * x = b$

In this numerical example A is a real matrix such that $a_{ij} = i$, if $i = j$, else $a_{ij} = 10^{-|i-j|}$, $i, j = 1, .., N$, $N = 10$. Assume that b is a stochastic vector such that the component b_i is generated with a gaussian generator with a mean value $\sum_{j=1}^{n} a_{ij}$ and a standard deviation equal to $1.e - 4 = 10^{-4}$. With such kind of system, the solutions x_i are around 1.

The theoretical standard deviations on each component of the solution are obtained according the method described in the previous section. First the matrix D is computed from matrix A. Then the system $y = D^{-1}c$ is solved, and

Table 1. Percentages of theoretical standard deviation outside the confidence interval

N \ k	3	4	5	6	7	10
10	12.1	6.3	3.3	2.1	1.5	0.3
100	12.6	5.3	3.8	2.3	1.0	0.3
1000	13.2	4.6	3.9	1.6	1.4	0.2
10000	11.6	5.4	2.9	1.9	1.4	0.2

Table 2. Theoretical and computed standard deviations

Component i	Theoretical standard deviation x''	mean value of the computed standard deviations
1	9.98e-05	10.4e-05
2	4.97e-05	4.06e-05
3	3.32e-05	3.21e-05
4	2.49e-05	2.02e-05
5	1.99e-05	1.81e-05
6	1.66e-05	1.50e-05
7	1.42e-05	1.54e-05
8	1.24e-05	1.02e-05
9	1.11e-05	0.778e-05
10	0.999e-05	0.806e-05

the standard deviations are computed with the formula $x''_i = \tau(y_i)\sqrt{|y_i|}$. The values x''_i are given in the first column in Table 2.

The experimental results only concern the standard deviations on the components of the solution. They are obtained in the following way: 30 different vectors $b^{(k)}, k = 1, ..., 30$ and thus 30 systems $A * x = b^{(k)}$ are generated as above. Then they are solved using the CADNA software using Gaussian elimination. This CADNA software provides the standard deviation of each component the solution. In the end, the mean value of the standard deviations of the 30 samples are computed for the $N = 10$ components and printed in Table 2. As we can see in Table 2, the theoretical standard deviations and the computed values are very close to each other.

The influence of the variation of the error σ on the right hand side b on the results is studied as follows: The same as above procedure is performed but here only the first component of the solution is considered. As before, 30 solutions for

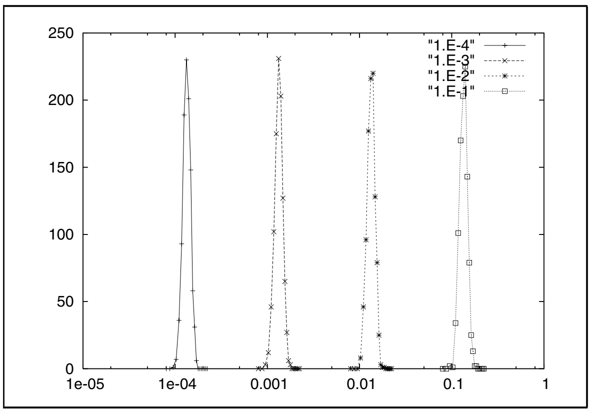

Fig. 1.

x are obtained and the standard deviations of the 30 first components $x^{(1)}$ and their corresponding mean value named $\bar{\delta}_i(1)$ are computed. This procedure has been performed 1000 times. Therefore, we obtained 1000 values for $\bar{\delta}_i(1)$ which are classified in 20 classes from $0.5x_i''$ to $1.5x_i''$. The graphs of the number of elements in each class obtained with the 4 values of $\sigma\ = 1.e-4, 1.e-3, 1.e-2, 1.e-1$ are reported in Fig. 1.

5 Conclusion

The theoretic study of the properties of stochastic numbers with respect to the operations addition and multiplication by scalars allows the solution of certain algebraic problems involving stochastic numbers. This gives us a possibility to compare algebraically obtained results with practical applications of stochastic numbers, such as the ones provided by the CESTAC method [3]. Such comparisons give additional information related to the stochastic behaviour of random roundings in the course of numerical computations. It may be expected that the proposed theory can be used in the computational practice.

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References

1. Alt, R., Error Propagation in Fourier Transforms, Mathematics and Computers in Simulation XX (1978) 37–43.
2. Alt, R., S. Markov, On the Algebraic Properties of Stochastic Arithmetic. Comparison to Interval Arithmetic, In: W. Kraemer et al.(Eds.) Scientific Computing, Validated Numerics, Interval Methods, Kluwer, 2001, 331–341.
3. Alt, R., J. Vignes, Validation of Results of Collocation Methods for ODEs with the CADNA Library. *Appl. Numer. Math.* 20 (1996), 1–21.
4. Markov, S., R. Alt, Stochastic arithmetic: Addition and multiplication by scalars, *Appl. Numer. Math.* 50 (2004), 475–488.
5. Markov, S., R. Alt, Jean-Luc Lamotte; *Stochastic Arithmetic: S-spaces and Some Applications*, *Num. Algorithms*, 37(1-4), 275–284, 2004.
6. Parker, D. S., Monte Carlo Arithmetic: Exploiting randomness in floating point arithmetic, *Techn. Report, UCLA Computer Sci. Dept., January 1997.*
7. Vignes, J., R. Alt, An Efficient Stochastic Method for Round-Off Error Analysis, in: *Accurate Scientific Computations, LNCS 235*, Springer, 1985, 183–205.
8. Vignes, J., Review on Stochastic Approach to Round-Off Error Analysis and its Applications. *Math. and Comp. in Sim.* 30, 6 (1988), 481–491.
9. Vignes, J., A Stochastic Arithmetic for Reliable Scientific Computation, *Math. and Comp. in Sim.* 35 (1993), 233–261.